Problem 1. Given a vector $\mathbf{b} \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times m}$, the Arnoldi process is a systematic way of constructing an orthonormal bases for the sucessive Krylov subspaces

$$\mathcal{K}_n = \langle \mathbf{b}, A\mathbf{b}, \ldots, A^{n-1}\mathbf{b} \rangle, \quad n = 1, 2, \ldots.$$

It gives

$$AQ_n = Q_{n+1}\tilde{H}_n,$$

where $Q_n \in \mathbb{R}^{m\times n}$, $Q_{n+1} \in \mathbb{R}^{m\times(n+1)}$ are with orthonormal columns and $\tilde{H}_n \in \mathbb{R}^{(n+1)\times n}$ is upper-Hessenberg. Let $H_n \in \mathbb{R}^{n\times n}$ be obtained by deleting the last row of $\tilde{H}_n$.

(a) Write out the Arnoldi algorithm.

(b) Assume that at step n, the $(n+1, n)$-th entry of $\tilde{H}_n$ is zero.

  i. Show that $\mathcal{K}_n$ is an invariant subspace of A and that $\mathcal{K}_n = \mathcal{K}_{n+1} = \mathcal{K}_{n+2} = \ldots$.

  ii. Show that each eigenvalue of H_n is an eigenvalue of A for $n > 1$.

(c) Let P_n be the set of monic polynomials of degree n. Show that the minimizer of

$$\min_{p_n \in P_n} \|p_n(A)\mathbf{b}\|_2$$

is given by the characteristic polynomial of H_n.

Problem 2. The following FitzHugh-Nagumo model is a simplified version of the Hodgkin-Huxley model (1963 Nobel Prize in Physiology or Medicine) which models in a detailed manner activation and deactivation dynamics of a spiking neuron.

$$\epsilon \dot{v} = v - \tfrac{1}{3}v^3 - w + I_{\text{ext}}$$

$$\dot{w} = v + a - bw$$

where v is the membrane voltage, w is a linear recovery variable, I_{ext} is the external stimulus. It contains the van der Pol oscillator as a special case for $a = b = I_{\text{ext}} = 0$. Fig 1 gives a qualtitative description of the four-stage structure of FitzHugh-Nagumo limit cycle solution and Fig 2 sketchs the time-profile illustrating the four stages in the limit cycle.

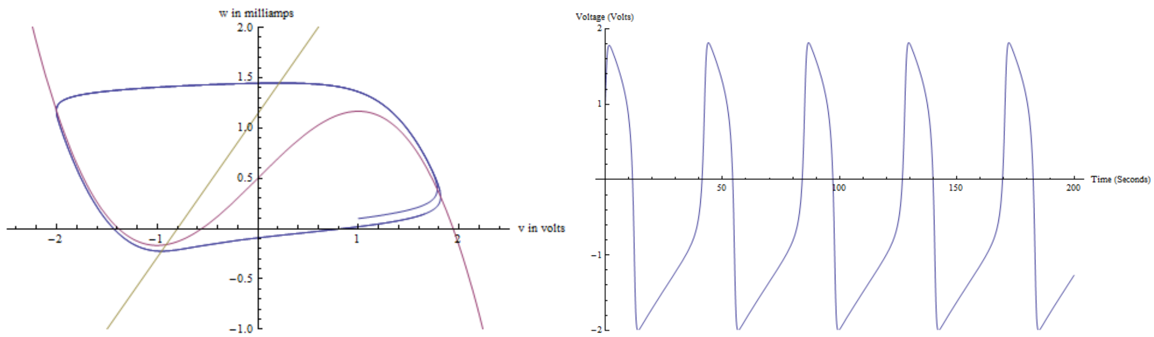

Figure 1: (a) The blue line is the trajectory of the FHN model in phase space. The pink line is the cubic nullcline $w = v - \frac{1}{3}v^3 + I_{\text{ext}}$ and the yellow line is the linear nullcline $w = a/b + v/b$. (b) Graph of v with parameters $I_{\text{ext}} = 0.5, a = 0.7, b = 0.8$, and $\epsilon = 1/12.5$.

(a) (10 pts) Use the expansions $v(t) = v_0(t) + \epsilon v_0(t) + O(\epsilon^2)$, $w(t) = w_0(t) + \epsilon w_0(t) + O(\epsilon^2)$ to determine the equations for the leading order slow solution. Point out the slow manifold (w_0 as a function of v_0) in Fig 1(a) and indicate the direction of the motion on each part, and identify the two attracting points on the curve.

(b) (5 pts) Use the expansion $v(t) = V_0(T) + \epsilon V_1(T) + O(\epsilon^2)$, $w(t) = W_0(T) + \epsilon W_1(T) + O(\epsilon^2)$ with $T = t/\epsilon$ to obtain the equations for the leading order fast solution.

(c) (5 pts) Use the phase plane to determine the maximum and the minimum values of $v(t)$ during an oscillation. Point out in Fig 1(a) and Fig 1 (b) the part of slow dynamics and fast dynamics, estimate the period for $v(t)$ as a function of time.